Language models for statisticians:
from *n*-grams to transformers to chatbots

Bob Carpenter

Center for Computational Mathematics
Flatiron Institute

July 2023

FLATIRON
INSTITUTE

What is a language model?

- **Language** uses a **finite** number of symbols called **tokens**
  - we assume a finite **token set** Tok of size K
- Tokens may be letters, words, sounds, syllables, etc.
  - GPT uses **sequences of letters** (average 1.5 tokens per English word)
- Treat language as a **stochastic process**
  - $Y = Y_1, Y_2, \ldots$ for random variables $Y_n \in \text{Tok}$
- Models typically **autoregressive**, predicting next word from previous

N-gram language models **(Shannon 1948)**

- Assume language process is **order-*N* Markov**
  - tokens conditionally independent given previous $N-1$ tokens

$$p(y_k \mid y_{k-1}, \ldots, y_1) = p(y_k \mid \underbrace{y_{k-1}, \ldots, y_{k-N-1}}_{N-1 \text{ tokens}}).$$

- Even **GPT is Markovian**
  - GPT-3: $N = 4096$ GPT-4: $N = 8192$ Claude: $N = 100,000$
  - **bottleneck** is $\mathcal{O}(N^2)$ attention algorithm (Claude more clever?)
  - cf. a real computer is technically a finite-state machine

Shannon's *N*-gram models

- **Claude Shannon**. 1948. **A Mathematical Theory of Communication.** *Bell System Technical Journal*.
- Shannon used English **letters** ($K = 1, 2, 3$) and **words** ($K = 1, 2$)
- **What is English?** How do we collect a **sample**?
- Shannon used **books of frequencies**
  - **letter trigrams** (1939 book); **word bigrams** (1923 book)
- Fit and inference usually **regularized MLE** for efficiency
  - ensures **non-zero probability** for any sequence

Shannon's fit

- MLE probabilities from compiled tables of letters (1923), words (1939)
  - or, open books at random, find current context, generate following word
- Shannon generated random examples
  - **Order 1, letters**: OCRO HLI RGWR NMIELWIS EU LL NBNESEBYA TH EEI ALHENHTTPA OOBTTVA NAH BRL.
  - **Order 3, letters**: IN NO IST LAT WHEY CRATICT FROURE BIRS GROCID PONDENOME OF DEMONSTURES OF THE REPTAGIN IS REGOACTIONA
  - **Order 1, words**: REPRESENTING AND SPEEDILY IS AN GOOD APT OR COME CAN DIFFER-ENT NATURAL HERE HE THE A IN CAME THE TO
  - **Order 2, words**: THE HEAD AND IN FRONTAL ATTACK ON AN ENGLISH WRITER THAT THE CHARACTER OF THIS POINT IS THEREFORE

Measuring accuracy with entropy

- Accuracy of N-gram language model p_Y measured with **entropy (rate)**
- Given a random sequence $Y \in \mathsf{Tok}^K$, its **entropy** in **bits** (base 2) is

$$\mathrm{H}[Y] \;=\; \mathbb{E}[\log_2 p_Y(Y)] \;=\; \sum_{y \in \mathsf{Tok}^K} p_Y(y) \cdot \log_2 p_Y(y).$$

- The **entropy rate** is average entropy per token, $\lim_{K \to \infty} \mathrm{H}[Y]/K$,
- The entropy rate for N-grams is given by **conditional entropy**,

$$\mathrm{H}[Y_K \mid Y_{K-1}, \ldots, Y_{K-N-1}] \;=\; \mathbb{E}[\,\log_2 p(Y_K \mid Y_{K-1}, \ldots, Y_{K-N-1})\,]$$

Signal processing: entropy and compression

- Shannon (1948) introduced **information theory** to model **signal compression** and decompression for communication
- Assume a language model with pmf p_Y
- **Compress** $y \in \mathsf{Tok}^*$, to $\lceil \log_2 p_Y(y) \rceil$ bits
  - in practice with **arithmetic coding** (Witten, Neal, Cleary 1987)

OpenAI's GPT-3: Published

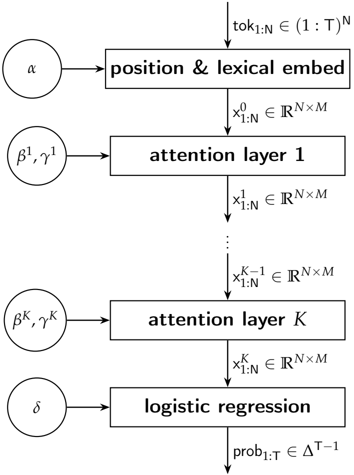
- **Training set** sizes

Source	Tokens
Common Crawl	410 billion
Books2	55 billion
WebText2	19 billion
Books1	12 billion
Wikipedia	3 billion
	$\approx$ 500 billion

- **Number of parameters**: $\approx$175 billion
- **Context history size**: 4K tokens
- Let's turn to **how it works . . .**

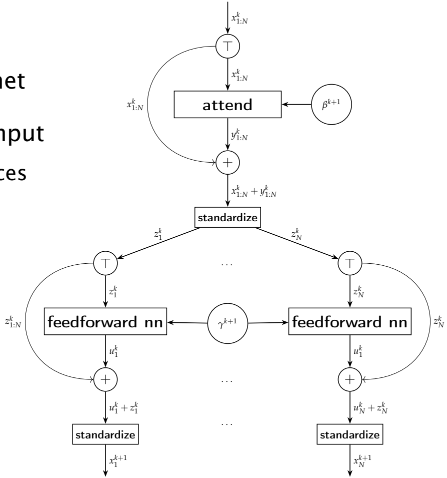
Top-level architecture

- **Transformer**, but **decoder only**
- tok_n: n-th **input token**
- x_n^k: **value of token** n at layer k
- prob_t: **probability next token** is t
- **circles** enclose model parameters



Attention architecture

- attention then feedforward neural net
- **ResNet architecture**: tees to add input
  - ≈ **hierarchical model** of differences
  - non-centered parameterization
- standard two-layer **neural nets**
  - **shared params** for each value
- **standardized** for numerics



Pseudocode: GPT in 40 lines

```
SIZES
------------------------------------
T:  number of distinct tokens
N:  size of context (history)

V:  size of token embedding vectors
A:  number of attention layers
K:  size of keys and queries
L:  width of neural network
```

```
DECODE(tok:  int<low=1,up=T>[N],    alpha:  matrix(T, V),
       betas: { query:matrix(K, V),
                key:matrix(K, V),
                value: matrix(V, V) }[A]
       gammas:  nn(V, L)[A],
       delta:   {1: vector[T],
                 2: matrix(T, N * V)}): simplex[T]
-----------------------------------------------------------
for n in 1:N:
    xs[0, n] = LEX(tok[n], alpha) + POS(n)
for a in 1:A:
    xs[a] = ATTEND(xs[a - 1], betas[a], gammas[a])
    for n in 1:N:
        xs[a, n] = FEED_FORWARD(xs[a, n], gammas[a])
y = STANDARDIZE(delta.1 + delta.2 * xs[A].flatten())
return SOFTMAX(y)
```

```
LEX(t:      int<low=1,up=T>,
    alpha:  vector(V)[T]):  vector(V)
----------------------------------------------------------
return alpha[t]


POS(n:  int<low=1,up=N>):  vector(V)
----------------------------------------------------------
for i in 1:V / 2:
    r = n / N**(2 * i / V)      // pos / max_pos^(0..2]
    u[2 * i] = sin(r)
    u[2 * i + 1] = cos(r)
return u
```

```
ATTEND(x:      vector(V)[N],
       beta:   { query: matrix(K, V), key: matrix(K, V),
                 value: matrix(V, V)},
       gamma:  nn(V, L)):  vector(V)[N]
------------------------------------------------------------
for n in 1:N:
    q[n] = beta.query * x[n]
    k[n] = beta.key * x[n]
    v[n] = beta.value * x[n]
for n in 1:N:
    lp[1:n-1] = [q[n]' * k[1], ..., q[n]' * k[n-1]] / sqrt(V)
    lp[n:N] = -inf
    p = SOFTMAX(lp)
    u[n] = SUM(n in 1:N) p[n] * v[n]
    y[n] = STANDARDIZE(u[n] + x[n])
return y
```

```
FEED_FORWARD(x:     real[R],
             alpha: { 1: real[S], 2: real[S, R],
                      3: real[R], 4: real[R, S]):  real[R]
-----------------------------------------------------------
u = alpha.1 + alpha.2 * x
v = GELU(u)
y = alpha.3 + alpha.4 * v
return STANDARDIZE(x + y)

GELU(v:  real[R]):  real[R]
    return [v_i * Phi(v_i) for v_i in v]

STANDARDIZE(v:  real[R]):  real[R]
    return (v - mean(v)) / std_dev(v)

SOFTMAX(real[R] v):  simplex(R)
    return exp(v) / sum(exp(v))
```

Multi-head attention

- What we have presented is **single-head attention**
- In practice, GPT uses **multi-head attention**
  - J parallel attention “heads”
  - keys, values, queries for each head **projected from previous layer value**
  - **value projected** for each head down to size to V/J
  - **concatenate** output of each head to produce size V value
- GPT-4 rumored to use parallel GPTs in an ensemble

GPT-3 sizes

- 175 billion parameters
- 96 layers
- 12,288 total value width
- 96 parallel attention heads
  - 128 value width per head

From LLM to Chatbot

- **LLM goal**: predict **next token on web** page
- **Chatbot goal** is to train a model that is
  - **helpful**: help users solve task
  - **honest**: shouldn't fabricate or mislead user
  - **harmless**: shouldn't cause physical, psychological, social, or environmental harm
- Strategy is to **align** an LLM to be a Chatbot with **fine tuning**
  - LLM acts as an **informative prior**
  - In ML terms, LLM provides **inductive bias**

Reinforcement learning with human feedback (RLHF)

1. Supervised fine tuning
   - human raters **provide desired output** for sampled prompts
   - **fine-tune** LLM with **supervised learning**
2. Reward model training
   - human raters **rank multiple outputs** for sample prompts
   - train a **reward model**
3. Reinforcement learning
   - **policy ranks outputs** for sample prompts
   - fine-tune LLM with **proximal policy optimization** (PPO)

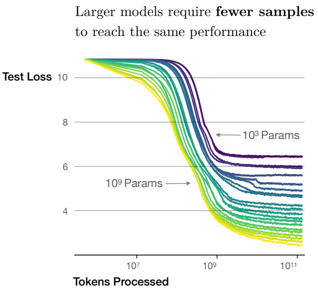
Some caveats (OpenAI 2022)

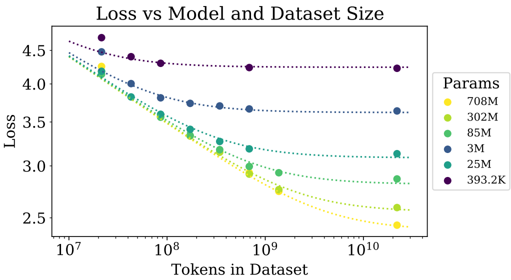
- "This procedure aligns the behavior of GPT-3 to the **stated preferences of a specific group** of people (mostly our labelers and researchers), rather than to any **broader notion of "human values"**.
  - cf. **Cultural consensus theory** provides mixture model of "values"

- "During RLHF fine-turning, we observe **performance regressions** compared to GPT-3 on certain public NLP datasets.
  - i.e., performance degrades relative to unaligned model
  - partially mitigated by **hierarchical modeling** alternating reinforcement and supervision

Loss vs. tokens, model size (OpenAI)

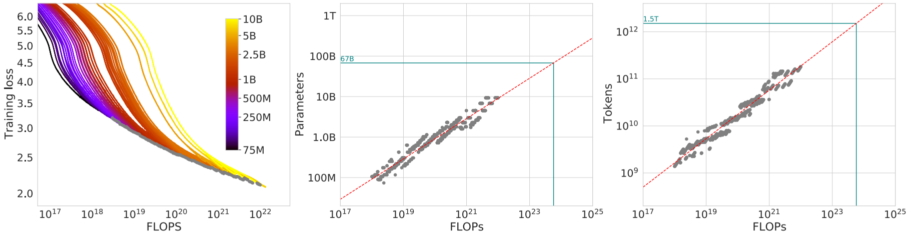
- Accuracy is **bounded by parameter size** (right)
- Accuracy is **bounded by data size** (left)





Scaling models (DeepMind)

- Accuracy **determined by flops**
  - for given **flops**, there is an **optimal choice** of **training tokens** and **model size**
  - fits held out predictions **very well**



  - (l) loss by model size, (c) optimal parameters, (r) optimal train tokens

OpenAI's GPT-4: Unpublished

- **Training set** unpublished (estimated ≈5 trillion)
- **Parameter set** unpublished (estimated ≈2 trillion)
- **Context history size**: 8K or 32K tokens
- **Cluster cost** training: ≈US$500 million (incl. 10K+ US$15K GPUs)
- **Marginal cost** training: ≈US$10s of millions (hardware, power, staff)
- **Open AI** is now **ClosedAI**: "Given both the competitive landscape and the safety implications of large-scale models like GPT-4, **this report contains no further details** about the architecture (including model size), hardware, training compute, dataset construction, training method, or similar."

The cat's out of the bag

- Transformer LLM architecture published by **Google** (2017)
- Alignment to ChatBots published by **OpenAI** (2022)
  - Meta (nee Facebook): **LLaMA**
    - **Open source** for research (since leaked)
    - Stanford CS: **Alpaca** fine-tuned
    - Runs 2 tokens/second on iMac with 4-bit floating point
  - Google: **Bard**
  - Google and OpenAI: **Copilot** (code/programming API integration)
  - Anthropic: **Claude** (100K token context) (branded as **Poe** for writing)
  - Many smaller, less widely used alternatives

LLM References

1. *The transformer paper*:

   Vaswani et al. (Google). 2017. (82K citations)
   Attention is all you need. *NeurIPS*.

2. *LLMs are highly generalizable*:

   Brown et al. (OpenAI). 2020. (12K citations)
   Language models are few-shot learners. *NeurIPS*.

3. *Going from GPT to ChatGPT*:

   Ouyang et al. (OpenAI). 2022. (1.5K citations)
   Training language models to follow instructions. *NeurIPS*.

4. *Original OpenAI paper on scaling*:

   Kaplan et al. 2020. (0.6K citations)
   Scaling laws for neural language models. *arXiv*

5. *Chinchilla paper on scaling laws for transformers*:

   Hoffmann et al. (DeepMind) 2022. (0.1K citations)
   Training compute-optimal large language models. *arXiv*

6. *What can GPT-4 do?*

   Bubeck et al. (Microsoft). 2023. (0.4K citations)
   Sparks of artificial general intelligence. *arXiv*.

7. *Another pseudocode I found after I did mine*:

   Phuong & Hutter (DeepMind). 2022. (0.02K citations)
   Formal algorithms for transformers. *arXiv*.

8. *Reproducible PyTorch case study with Colab notebook fitting Shakespeare*:

   Andrej Karpathy (now at OpenAI). 2023. (2.8M views)
   Let's build GPT: from scratch, in code, spelled out. YouTube!